Spatially-Coupled MacKay-Neal Codes Universally Achieve the Symmetric Information Rate of Arbitrary Generalized Erasure Channels with Memory

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Abstract—**This paper investigates the belief propagation decoding of spatially-coupled MacKay-Neal (SC-MN) codes over erasure channels with memory. We show that SC-MN codes with bounded degree universally achieve the symmetric information rate (SIR) of arbitrary erasure channels with memory. We mean by universality the following sense: the sender does not need to know the whole channel statistics but needs to know only the SIR, while the receiver estimates the transmitted codewords from channel statistics and received words. The proof is based on potential function.**

Index Terms—spatially-coupled codes, MacKay-Neal codes, LDPC codes, potential function

I. INTRODUCTION

Felström and Zigangirov introduced convolutional LDPC codes [1]. Later the codes are called spatially-coupled (SC) codes. Lentmaier et al. confirmed that regular SC LDPC codes achieve excellent BP decoding [2]. Kudekar et al. proved that SC codes achieve MAP threshold by BP decoding on the binary erasure channel (BEC) [3] and binary memoryless symmetric channels [4].

In [5], Takeuchi et al. introduced potential function to understand how threshold saturation phenomenon happen. With some modifications, Yedla et al. proved threshold saturation of spatially-coupled LDPC codes over BEC in [6], [7].

Kasai et al. introduced SC MacKay-Neal (MN) codes, and showed that these codes with finite maximum degree achieve capacity of BEC by numerical experiment [8]. Obata et al. [9] and Okazaki et al. [10] proved respectively, $(d_l, d_r = 2, d_r = 2)$ SC-MN codes and $(d_l, d_r = 3, d_g = 3)$ SC-MN codes achieve the capacity of BEC for any $d_l > d_r$, where each of d_l, d_r, d_g describes the degree of nodes in the Tanner graph of the codes.

Reliable transmissions over channels with memory are practically important, e.g., magnetic recording with ISI and Rayleigh fading channel with memory [11]. Pfister and Siegel proved that carefully designed irregular LDPC codes can achieve the symmetric information rate (SIR) of dicode erasure channels (DEC) under joint iterative decoding [12]. In [13] and [14], it was empirically observed that the BP threshold

of spatially-couple regular codes achieve the SIR of DEC and PR2 channels, respectively, by increasing node degree.

In this paper, we show that SC-MN codes with bounded degree universally achieve the symmetric information rate of wide variety of erasure channels with memory. We mean by universality the following sense: the sender does not need to know the whole channel statistics $p(\underline{y}|\underline{x})$ but needs to know only the SIR, while the receiver estimates the transmitted codewords from channel statistics and received words. The proof is based on the powerful potential function method [5], [6], [7].

II. BACKGROUND

A. Generalized Erasure Channel (GEC)

Denote a channel input and output as $\underline{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ and $\underline{y} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$, respectively. We assume that $\underline{x} \in \{0,1\}^n$ is uniformly distributed. Let us assume that by introducing some appropriate state nodes $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$ such that

$$p(\underline{x}|\underline{y}) = \sum_{\underline{\sigma}} p(\underline{x}, \underline{\sigma}|\underline{y}),$$

we can factorize $p(\underline{x}, \underline{\sigma}|\underline{y})$ so that its factor graph is a tree. By Bayes rule, we have $p(\underline{x}, \underline{\sigma}|\underline{y}) \propto p(\underline{y}|\underline{\sigma}, \underline{x})p(\underline{\sigma}|\underline{x})p(\underline{x})$. Consider the APP detector implemented by sum-product algorithm to calculate

$$\big(p(x_j = 0|\underline{y}), p(x_j = 1|\underline{y})\big) =: \big(\mu_j(0), \mu_j(1)\big)$$

for $j = 1, 2, \ldots, n$. We say the channel $p(\underline{y}|\underline{x})$ is a generalized erasure channel (GEC) if $(\mu_j(0), \mu_j(1))$ is one of $\{(1,0), (0,1), (1/2, 1/2)\}$. The corresponding LLR values are $+\infty$, $-\infty$ and 0. Some authors use 0, 1 and $?$, instead.

B. LDPC codes over GEC

Next, consider $\underline{x}$ is uniformly distributed in an LDPC code C. In precise,

$$p(\underline{x}) = \begin{cases} 1/\#C & (\underline{x} \in C) \\ 0 & (\underline{x} \notin C) \end{cases}$$

Let us denote the factor graph of $p(\underline{y}|\underline{\sigma},\underline{x})p(\underline{\sigma}|\underline{x})p(\underline{x})$ by G. We divide G into two subgraphs. One is the factor graph of $p(\underline{y}|\underline{\sigma},\underline{x})p(\underline{\sigma}|\underline{x})$ and the other is the factor graph of $p(\underline{x})$. They are corresponding to APP detector and LDPC decoder, respectively. Since the message from APP detector to the bit nodes is one of $\mathcal{M}$, the messages used in the LDPC decoder also takes value in $\mathcal{M}$.

Consider the density evolution of the BP decoding on G in the limit of code length. We define $\phi(x;\epsilon)$ as a function which maps the erasure probability of messages from LDPC decoder to APP detector via bit nodes x, to the erasure probability of messages from APP detector to from LDPC decoder $\phi(x;\epsilon)$, where ϵ is a parameter which defines the channel. From the definition, it follows that $\phi(x;\epsilon)$ is non-decreasing in $x \in [0,1]$. In this paper, we further assume that $\phi(x;\epsilon)$ is twice continuously differentiable with x and strictly increasing with $\epsilon \in [0,1]$. In this setting, the density evolution for (d_l, d_r) regular LDPC codes over GEC with $\phi(x;\epsilon)$ is written as follows.

$$x^{(0)} = 1,$$
$$x^{(t+1)} = \phi\big((1-(1-x^{(t)})^{d_r-1})^{d_l};\epsilon\big)$$
$$\cdot (1-(1-x^{(t)})^{d_r-1})^{d_l-1},$$

where $x^{(t)}$ is the erasure probability of messages from bit nodes to parity-check nodes at the t-th decoding round.

C. Symmetric Information Rate

SIR $I(\epsilon)$ is the mutual information is defined as follows

$$I(\epsilon) := \lim_{n\to\infty} \frac{1}{n} I(\underline{X}; \underline{Y}(\epsilon)),$$

under the existence of limit, where capital letters represent random variables. In [15], [16], it is shown that the SIR is calculated via $\phi(x;\epsilon)$ as follows.

$$I(\epsilon) = 1 - \int_0^1 \phi(x;\epsilon)dx = 1 - \Phi(1;\epsilon),$$

where $\Phi(x;\epsilon) := \int_0^x \phi(x';\epsilon)dx'$. Let R be the coding rate. Define SIR limit as ϵ such that $I(\epsilon) = R$, and denote it by $\epsilon^{\mathrm{SIR}}(R)$. Uniqueness of $\epsilon^{\mathrm{SIR}}(R)$ is again due to the assumption that $\phi(x;\epsilon)$ is increasing in ϵ.

D. MacKay-Neal Codes

(d_l, d_r, d_g) MN codes are multi-edge type (MET) LDPC used over GEC with $\phi(x;\epsilon)$ codes [17] defined by pair of multi-variables degree distributions (μ, ν) listed below.

$$\nu(\boldsymbol{x};\Phi(x;\epsilon)) = \frac{d_r}{d_l}x_1^{d_l} + \Phi(x_2^{d_g};\epsilon), \quad \mu(\boldsymbol{x}) = x_1^{d_r}x_2^{d_g}.$$

Here, we slightly extended the definition of degree distribution in such a way that the bits corresponding to the term $\Phi(x_2^{d_g};\epsilon)$ are transmitted through the GEC with ϕ_ϵ. In the case of BEC(ϵ), $\Phi(x,\epsilon) = \epsilon x$. We define the erasure probability message sent from bit nodes along edges of type j at the t-th

decoding round by $x_j^{(t)}$. The recursion of density evolution of MET-LDPC codes on BEC is given by

$$x_j^{(t)} = \frac{\nu_j(\boldsymbol{y}^{(t)};\Phi(x;\epsilon))}{\nu_j(\mathbf{1};\mathrm{id}_\mathbb{R})}, \quad y_j^{(t+1)} = 1 - \frac{\mu_j(\mathbf{1}-\boldsymbol{x}^{(t)})}{\mu_j(\mathbf{1})},$$

where $\nu_j(\boldsymbol{x};\Phi(x;\epsilon)) := \frac{\partial}{\partial x_j}\nu(\boldsymbol{x};\Phi(x;\epsilon))$, $\mu_j(\boldsymbol{x}) := \frac{\partial}{\partial x_j}\mu(\boldsymbol{x})$ and $\mathrm{id}_\mathbb{R}$ is the identity function $\mathrm{id}_\mathbb{R}(x) = x$.

Then, the density evolution of (d_l, d_r, d_g) MN codes is

$$\boldsymbol{x}^{(0)} = \mathbf{1}, \quad \boldsymbol{x}^{(t+1)} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}^{(t)});\epsilon),$$

where

$$\boldsymbol{f}(\boldsymbol{x};\epsilon) := \big(x_1^{d_l-1}, \phi(x_2^{d_g};\epsilon)\big) \tag{1}$$
$$\boldsymbol{g}(\boldsymbol{x}) := \big(1-(1-x_1)^{d_r-1}(1-x_2)^{d_g},$$
$$1-(1-x_1)^{d_r}(1-x_2)^{d_g-1}\big). \tag{2}$$

E. Spatially-Coupled MacKay-Neal Codes

SC-MN codes of chain length L and of coupling width w are defined by the Tanner graph constructed by the following process. First, at each section $i \in \mathbb{Z}$, place rM/l bit nodes of type 1 and M bits nodes of type 2. Bit nodes of type 1 and 2 are of degree d_l and d_g, respectively. Next, at each section $i \in \mathbb{Z}$, place M check nodes of degree $d_r + d_g$. Then, connect edges uniformly at random so that bit nodes of type 1 at section i are connected with check nodes at each section $i \in [i, \ldots, i+w-1]$ with $d_r M/w$ edges, and bit nodes of type 2 at section i are connected with check nodes at each section $i \in [i, \ldots, i+w-1]$ with $d_g M/w$ edges. Bits at section $i \notin [0, L-1])$ are shortened. Bits of type 1 and 2 at section $i \in [0, L-1]$ are punctured and transmitted, respectively. The rate of SC-MN codes $R^{MN}(d_l, d_r, d_g, L, w)$ is given by

$$R^{\mathrm{MN}}(d_l, d_r, d_g, L, w)$$
$$= \frac{d_r}{d_l} + \frac{1+w-2\sum_{i=0}^w(1-(\frac{i}{w})^{d_r+d_g})}{L}$$
$$= \frac{d_r}{d_l} \quad (L \to \infty). \tag{3}$$

F. Vector Admissible System and Potential Function

In this section, we define vector admissible systems and potential functions.

Definition 1. Define $\mathcal{X} \triangleq [0,1]^d$, and $F : \mathcal{X} \times [0,1] \to \mathbb{R}$ and $G : \mathcal{X} \to \mathbb{R}$ as functional satisfying $G(\mathbf{0}) = 0$. Let $\boldsymbol{D}$ be a $d \times d$ positive diagonal matrix. Consider a general recursion defined by

$$\boldsymbol{x}^{(\ell+1)} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}^{(\ell)});\epsilon) \tag{4}$$

where $\boldsymbol{f} : \mathcal{X} \times [0,1] \to \mathcal{X}$ and $\boldsymbol{g} : \mathcal{X} \to \mathcal{X}$ are defined by $F'(\boldsymbol{x};\epsilon) = \boldsymbol{f}(\boldsymbol{x};\epsilon)\boldsymbol{D}$ and $G'(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})\boldsymbol{D}$, where $F'(\boldsymbol{x};\epsilon) \triangleq (\frac{\partial F(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial F(\boldsymbol{x})}{\partial x_n})$. Then the pair $(\boldsymbol{f}, \boldsymbol{g})$ defines a vector admissible system if

1. $\boldsymbol{f}, \boldsymbol{g}$ are twice continuously differentiable,
2. $\boldsymbol{f}(\boldsymbol{x};\epsilon)$ and $\boldsymbol{g}(\boldsymbol{x})$ are non-decreasing in $\boldsymbol{x}$ and ϵ with respect to $\preceq$ [1],

[1] We say $\boldsymbol{x} \preceq \boldsymbol{y}$ if $x_i \le y_i$ for all $1 \le i \le d$

3. $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{0}); \epsilon) = \boldsymbol{0}$ and $F(\boldsymbol{g}(\boldsymbol{0}); \epsilon) = 0$.

We say $\boldsymbol{x}$ is a fixed point if $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}); \epsilon)$.

Definition 2 ([18, Def. 2]). We define the potential function $U(\boldsymbol{x}; \epsilon)$ of a vector admissible system $(\boldsymbol{f}, \boldsymbol{g})$ by

$$U(\boldsymbol{x}; \epsilon) \triangleq \boldsymbol{g}(\boldsymbol{x}) \boldsymbol{D} \boldsymbol{x}^T - G(\boldsymbol{x}) - F(\boldsymbol{g}(\boldsymbol{x}); \epsilon).$$

Definition 3 ([18, Def. 7]). Let $\mathcal{F}(\epsilon) \triangleq \{\boldsymbol{x} \in \mathcal{X} \setminus \{\boldsymbol{0}\} \mid \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}); \epsilon)\}$ be a set of non-zero fixed points for $\epsilon \in [0, 1]$. The potential threshold ϵ^* is defined by

$$\epsilon^* \triangleq \sup\{\epsilon \in [0, 1] \mid \min_{\boldsymbol{x} \in \mathcal{F}(\epsilon)} U(\boldsymbol{x}; \epsilon) > 0\}.$$

Let ϵ_s^* be threshold of uncoupled system defined in [18, Def. 6]. For ϵ such that $\epsilon_s^* < \epsilon < \epsilon^*$, we define energy gap $\Delta E(\epsilon)$ as

$$\Delta E(\epsilon) \triangleq \max_{\epsilon' \in [\epsilon, 1]} \inf_{\boldsymbol{x} \in \mathcal{F}(\epsilon')} U(\boldsymbol{x}; \epsilon').$$

Definition 4 ([18, Def. 9]). For a vector admissible system $(\boldsymbol{f}, \boldsymbol{g})$, we define the SC system of chain length L and coupling width w as

$$\boldsymbol{x}_i^{(t+1)} = \frac{1}{w} \sum_{k=0}^{w-1} \boldsymbol{f}\left(\frac{1}{w} \sum_{j=0}^{w-1} \boldsymbol{g}(\boldsymbol{x}_{i+j-k}^{(t)}); \epsilon_{i-k}\right),$$

$$\epsilon_i = \begin{cases} \epsilon, & i \in \{0, \ldots, L-1\}, \\ 0, & i \notin \{0, \ldots, L-1\}. \end{cases}$$

If we define $(\boldsymbol{f}, \boldsymbol{g})$ as the density evolution for (d_l, d_r, d_g) MN codes in (II-D) and (II-D), the SC system gives the density evolution of SC-MN codes with chain length L and coupling width w.

Next theorem asserts that if $\epsilon < \epsilon^*$ then fixed points of SC vector system converge towards $\boldsymbol{0}$ for sufficiently large w.

Theorem 1 ([18, Thm. 1]). Consider the constant $K_{\boldsymbol{f}, \boldsymbol{g}}$ defined in [18, Lem. 11]. This constant value depends only on $(\boldsymbol{f}, \boldsymbol{g})$. If $\epsilon < \epsilon^*$ and $w > (dK_{\boldsymbol{f}, \boldsymbol{g}})/(2\Delta E(\epsilon))$, then the SC system of $(\boldsymbol{f}, \boldsymbol{g})$ with chain length L and coupling width w has a unique fixed point $\boldsymbol{0}$.

It can be seen that the density evolution $(\boldsymbol{f}, \boldsymbol{g})$ of (d_l, d_r, d_g) MN codes over $\mathrm{GEC}(\phi(x; \epsilon))$ is a vector admissible system by choosing $F(\boldsymbol{x}; \epsilon), G(x)$ and $\boldsymbol{D}$ as below, since this system $(\boldsymbol{f}, \boldsymbol{g})$ satisfies the condition in Definition 1.

$$F(\boldsymbol{x}; \epsilon) = \frac{d_r}{d_l} x_1^{d_l} + \Phi(x_2^{d_g}; \epsilon),$$

$$G(\boldsymbol{x}) = d_r x_1 + d_g x_2 + (1 - x_1)^{d_r}(1 - x_2)^{d_g} - 1,$$

$$\boldsymbol{D} = \begin{pmatrix} d_r & 0 \\ 0 & d_g \end{pmatrix}.$$

From Definition 2, the potential function $U(x_1, x_2, \epsilon)$ of (d_l, d_r, d_g) MN codes is given by

$$U(x_1, x_2; \epsilon)) \tag{5}$$
$$= 1 - \Phi(\{1 - (1 - x_1)^{d_r}(1 - x_2)^{d_g-1}\}^{d_g}; \epsilon)$$
$$- \frac{d_r}{d_l}\{1 - (1 - x_1)^{d_r-1}(1 - x_2)^{d_g}\}^{d_l}$$
$$- (1 - x_1)^{d_r}(1 - x_2)^{d_g} \cdot \left(1 + \frac{d_r x_1}{1 - x_1} + \frac{d_g x_2}{1 - x_2}\right).$$

III. PROOF OF ACHIEVING SIR

In this section, we will prove that $(d_l, d_r = 2, d_g = 2)$ and $(d_l, d_r = 3, d_g = 3)$ SC-MN codes, for any $d_l > d_r$, achieve the SIR of any GEC. First, we calculate the potential threshold $\epsilon^*(d_l, d_r, d_g)$ of MN codes, and show that the potential threshold equals to the SIR limit $\epsilon^{\mathrm{SIR}}(d_r/d_r)$. Then we apply Theorem 1 which proves that density evolution of SC-MN code has a unique fixed point $\boldsymbol{0}$ for ϵ smaller than SIR limit $\epsilon^{\mathrm{SIR}}(d_r/d_r)$.

A. Potential Function at Trivial Fixed Point

Recall the definition of potential threshold $\epsilon^*(d_l, d_r, d_g)$ in Definition 3. We need to investigate the structure of $\mathcal{F}(\epsilon)$ to calculate the potential threshold $\epsilon^*(d_l, d_r, d_g)$. The density evolution (1) of (d_l, d_r, d_g) MN codes at fixed point $(x_1, x_2; \phi(x; \epsilon))$ can be rewritten as

$$x_1 = (1 - (1 - x_1)^{d_r-1}(1 - x_2)^{d_g})^{d_l-1}, \tag{6}$$

$$x_2 = \phi(\{1 - (1 - x_1)^{d_r}(1 - x_2)^{d_g-1}\}^{d_g}; \epsilon)$$
$$\cdot \{1 - (1 - x_1)^{d_r}(1 - x_2)^{d_g-1}\}^{d_g-1}. \tag{7}$$

First, observe that $(x_1 = 1, x_2 = \phi(1; \epsilon)) \in \mathcal{F}(\epsilon)$ for $\epsilon \in [0, 1]$. We call these fixed points trivial. From (II-F) and the definition of SIR limit, the next lemma asserts that the sign of $U(1, \phi(1; \epsilon); \epsilon)$ changes at the SIR limit $\epsilon^{\mathrm{SIR}}\left(\frac{d_r}{d_l}\right)$.

Lemma 1.

$$U(1, \phi(1; \epsilon); \epsilon) = 1 - \frac{d_r}{d_l} - \Phi(1; \epsilon),$$

$$U(1, \phi(1; \epsilon); \epsilon) \begin{cases} > 0 \text{ if } \epsilon < \epsilon^{\mathrm{SIR}}\left(\frac{d_r}{d_l}\right), \\ = 0 \text{ if } \epsilon = \epsilon^{\mathrm{SIR}}\left(\frac{d_r}{d_l}\right), \\ < 0 \text{ if } \epsilon > \epsilon^{\mathrm{SIR}}\left(\frac{d_r}{d_l}\right). \end{cases} \tag{8}$$

Proof: The first part is straightforward from (II-F). The second part is obvious from the definition of SIR limit. □

B. Potential Function at Non-Trivial Fixed Point

Next, for given $x_1 \in (0, 1)$, solve (III-A) in terms of x_2. Denote this by $x_2[x_1]$.

$$x_2[x_1] = 1 - \left(\frac{1 - x_1^{\frac{1}{d_l-1}}}{(1 - x_1)^{d_r-1}}\right)^{\frac{1}{d_g}}.$$

Note that for some x_1, $x_2[x_1]$ may fall outside $[0, 1]$ as one can see from Fig. 2. Such points are excluded from the set of fixed points. Then it follows that

$$(x_1, x_2[x_1]) \in \mathcal{F}(\epsilon[x_1]) \text{ iff } (x_2[x_1], \epsilon[x_1]) \in [0, 1]^2, \tag{9}$$

where $\epsilon[x_1]$ is the unique solution of the equation (III-A) with $x_2 = x_2[x_1]$ holds. In precise, the equation is as follows.

$$x_2[x_1] = \phi(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g-1}\}^{d_g}; \epsilon[x_1])$$
$$\cdot \{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g-1}\}^{d_g-1}.$$

Uniqueness is due to the assumption that $\phi(x; \epsilon)$ is strictly increasing in ϵ. We call such fixed points (III-B) non-trivial.
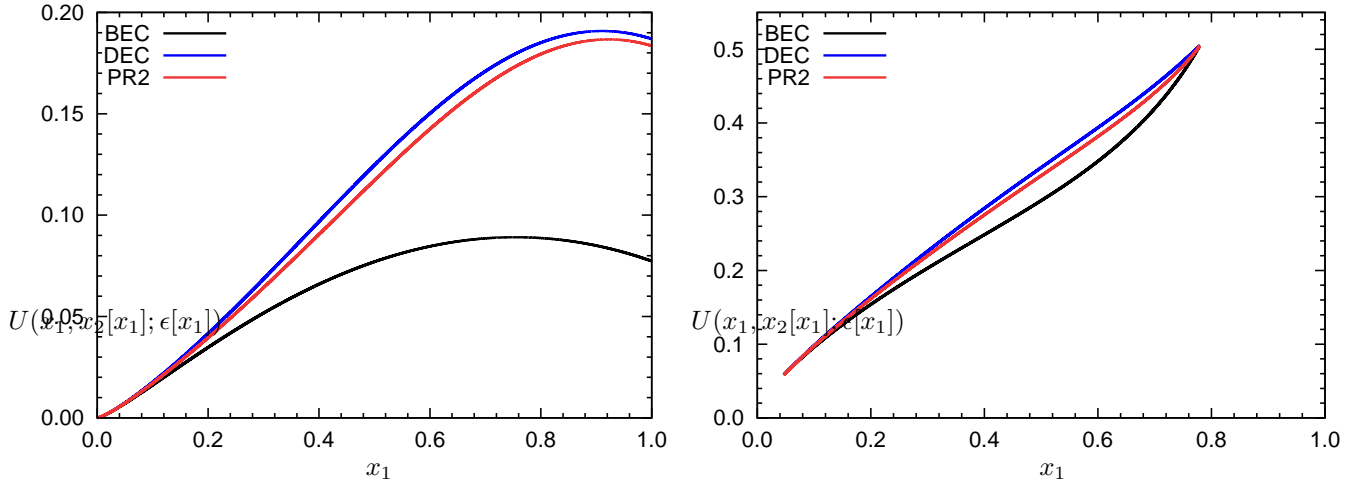
Fig. 1. Potential functions $U(x_1, x_2[x_1]; \epsilon[x_1])$ at non-trivial fixed points of $(4,2,2)$ MN codes and $(6,3,3)$ MN codes for 3 types of generalized erasure channels $\text{BEC}(\epsilon[x_1])$, $\text{DEC}(\epsilon[x_1])$, $\text{PR2}(\epsilon[x_1])$ and $\text{PR3}(\epsilon[x_1])$.

For example of $\text{BEC}(\epsilon)$, $\epsilon_{\text{BEC}}[x_1]$ can be written in an explicit form.

$$\epsilon_{\text{BEC}}[x_1] := \frac{x_2[x_1]}{\{1 - (1-x_1)^{d_r}(1-x_2[x_1])^{d_g-1}\}^{d_g-1}}. \quad (10)$$

Obviously, trivial and non-trivial fixed-points cover the set $\mathcal{F}(\epsilon)$, in precise, $\mathcal{F}(\epsilon) = \mathcal{F}_{\text{t}}(\epsilon) \cup \mathcal{F}_{\text{n}}(\epsilon)$. We denote the set of trivial and non-trivial fixed-points respectively by $\mathcal{F}_{\text{t}}(\epsilon)$ and $\mathcal{F}_{\text{n}}(\epsilon)$.

$$\mathcal{F}_{\text{t}}(\epsilon) := \big\{ \big(1, \phi(1;\epsilon)\big) \big\}$$
$$\mathcal{F}_{\text{n}}(\epsilon) := \big\{ (x_1, x_2[x_1]) \mid \epsilon[x_1] = \epsilon, x_1 \in (0,1) \big\}$$
$$\mathcal{F}_{\text{t}}(\epsilon) \cap \mathcal{F}_{\text{n}}(\epsilon) \text{ is an empty set.}$$

Let $\phi[x_1]$ be the unique solution of (III-A) in terms of $\phi(\cdot)$ for given $x_1, x_2 = x_2[x_1]$ and $\epsilon = \epsilon[x_1]$.

$$\phi[x_1] := \frac{x_2[x_1]}{\big(1 - (1-x_1)^{d_r}(1-x_2[x_1])^{d_g-1}\big)^{d_g-1}}. \quad (11)$$

Equivalently, we have

$$\phi[x_1] = \phi(\psi[x_1]; \epsilon[x_1]), \quad (12)$$

where

$$\psi[x_1] := \big(1 - (1-x_1)^{d_r}(1-x_2[x_1])^{d_g-1}\big)^{d_g}. \quad (13)$$

Note that $x_2[x_1], \phi[x_1]$ and $\psi[x_1]$ do not depend on the channel $\text{GEC}(\phi(\cdot;\epsilon))$, while $\epsilon[x_1]$ does. From (III-B) and (III-B), it can be seen that

$$\epsilon_{\text{BEC}}[x_1] = \phi[x_1]. \quad (14)$$

Substituting $x_1, x_2[x_1]$ and $\phi[x_1]$ into (II-F), at non-trivial fixed points $(x_1, x_2[x_1]) \in \mathcal{F}(\epsilon[x_1])$, we have the potential function as

$$U(x_1, x_2[x_1]; \epsilon[x_1]) \quad (15)$$
$$= 1 - \Phi(\psi[x_1]; \epsilon[x_1])$$
$$- \frac{d_r}{d_l}\{1 - (1-x_1)^{d_r-1}(1-x_2[x_1])^{d_g}\}^{d_l}$$
$$- (1-x_1)^{d_r}(1-x_2[x_1])^{d_g} \cdot \Big(1 + \frac{d_r x_1}{1-x_1} + \frac{d_g x_2[x_1]}{1-x_2[x_1]}\Big).$$

From [9, Lemma 4] and [10, Lemma 1, Lemma2], we have the following lemma.

Lemma 2. Consider transmissions over $\text{BEC}(\epsilon)$, i.e., we have $\phi(x) = \phi_{\text{BEC}}(x) = \epsilon$ and $\phi[x_1] = \epsilon[x_1]$. Let $U_{\text{BEC}}(x_1, x_2[x_1]; \epsilon_{\text{BEC}}[x_1])$ be the potential function of (d_l, d_r, d_g) MN codes at the non-trivial fixed points. For any $d_l > d_r$ it holds that

$$U_{\text{BEC}}(x_1, x_2[x_1]; \epsilon_{\text{BEC}}[x_1]) > 0 \text{ for } x_1 \in (0,1)$$

for $(d_r, d_g) = (2,2)$ and $(3,3)$.

Lemma 3. For any $\text{GEC}(\phi(x;\epsilon))$, let $U(x_1, x_2[x_1]; \epsilon[x_1])$ be the potential function at the non-trivial fixed point $(x_1, x_2[x_1]) \in \mathcal{F}(\epsilon[x_1])$ for $x_1 \in (0,1)$ as defined in (III-B). Then, it holds that

$$U(x_1, x_2[x_1]; \epsilon[x_1]) \geq U_{\text{BEC}}(x_1, x_2[x_1]; \epsilon_{\text{BEC}}[x_1]).$$

Proof: From (III-B), we have $(x_2[x_1], \epsilon[x_1]) \in [0,1]^2$. From (III-B), we have $\psi[x_1] \in [0,1]$. From this and using (III-B) and (III-B), we obtain $\epsilon_{\text{BEC}}[x_1] \in [0,1]$. This justifies that $U_{\text{BEC}}(x_1, x_2[x_1]; \epsilon_{\text{BEC}}[x_1])$ is well-defined since $x_1, x_2[x_1], \epsilon_{\text{BEC}}[x_1] \in [0,1]$. We have

$$U(x_1, x_2[x_1]; \epsilon[x_1]) - U_{\text{BEC}}(x_1, x_2[x_1]; \epsilon_{\text{BEC}}[x_1])$$

$$\overset{(a)}{=} \epsilon_{\text{BEC}}[x_1] \cdot (1 - (1-x_1)^{d_r}(1-x_2[x_1])^{d_g-1})^{d_g}$$
$$\quad - \Phi\big((1 - (1-x_1)^{d_r}(1-x_2[x_1])^{d_g-1})^{d_g}; \epsilon[x_1]\big)$$

$$\overset{(b)}{=} \phi[x_1] \cdot \psi[x_1] - \Phi\big(\psi[x_1]; \epsilon[x_1]\big)$$

$$= \phi[x_1] \cdot \psi[x_1] - \int_0^{\psi[x_1]} \phi(x'; \epsilon[x_1]) dx'$$

$$\overset{(c)}{=} \int_0^{\psi[x_1]} \phi[x_1] - \phi(x'; \epsilon[x_1]) dx'$$

$$\overset{(d)}{=} \int_0^{\psi[x_1]} \phi(\psi[x_1]; \epsilon[x_1]) - \phi(x'; \epsilon[x_1]) dx'$$

$$\overset{(e)}{\geq} 0.$$

The equality (a) is due to (III-B) and (1). The equality (b) is due to (III-B). In (c), we used the fact that $\psi[x_1]$ does

not depend on channel $\mathrm{GEC}(\phi(\cdot;\epsilon))$. The equality (d) is due to (III-B). In (e), we used the fact that $\phi(\psi[x_1];\epsilon[x_1]) \geq \phi(x';\epsilon[x_1])$ for $x' \in [0,\psi[x_1]]$ since $\phi(x;\epsilon)$ is non-decreasing in x. The equality is attained with $\phi(x;\epsilon) = \phi_{\mathrm{BEC}}(x;\epsilon) = \epsilon$. $\qquad\square$

In Fig. 2, we plotted the potential functions at non-trivial fixed points of $(4,2,2)$ MN codes and $(6,3,3)$ MN codes for 3 types of generalized erasure channels $\mathrm{BEC}(\epsilon[x_1])$, $\mathrm{DEC}(\epsilon[x_1])$ and $\mathrm{PR2}(\epsilon[x_1])$. One can see that the potential function for BEC is lower than other curves for any $x_1 \in (0,1)$ as claimed in Lemma 3.

C. Potential Threshold Equals to SIR Limit

Next theorem shows the potential threshold of some MN codes is equal to the SIR limit.

Theorem 2. For any $\mathrm{GEC}(\phi(x;\epsilon))$, the potential threshold ϵ^* of (d_l, d_r, d_g) MN codes is equal to $\epsilon^{\mathrm{SIR}}(\frac{d_r}{d_l})$ for any $d_l > d_r$, $(d_r, d_g) = (2,2)$ and $(3,3)$.

Proof: From Lemma 2 and Lemma 3, we have that for any $x_1 \in (0,1)$ such that $\epsilon = \epsilon[x_1] \in \left[0, \epsilon^{\mathrm{SIR}}(\frac{d_r}{d_l})\right)$,

$$U(x_1, x_2[x_1]; \epsilon[x_1]) > 0. \qquad (16)$$

It follows that

$$
\begin{aligned}
\epsilon^* &= \sup\{\epsilon \in [0,1] \mid \min_{(x_1,x_2)\in\mathcal{F}(\epsilon)} U(x_1,x_2;\epsilon) > 0\} \\
&= \sup\{\epsilon \in [0,1] \mid \min_{(x_1,x_2)\in\mathcal{F}_{\mathrm{t}}(\epsilon)\cup\mathcal{F}_{\mathrm{n}}(\epsilon)} U(x_1,x_2;\epsilon) > 0\} \\
&\overset{(a)}{=} \sup\{\epsilon \in \left[0, \epsilon^{\mathrm{SIR}}(\tfrac{d_r}{d_l})\right) \mid \min_{(x_1,x_2)\in\mathcal{F}_{\mathrm{t}}(\epsilon)\cup\mathcal{F}_{\mathrm{n}}(\epsilon)} U(x_1,x_2;\epsilon) > 0\} \\
&\overset{(b)}{=} \epsilon^{\mathrm{SIR}}\left(\tfrac{d_r}{d_l}\right).
\end{aligned}
$$

In (a), we used (1). In (b), we used (1) and (III-C). $\qquad\square$

Define the BP threshold $\epsilon^{\mathrm{BP}}(d_l, d_r, dg, L, w)$ as the spremum value of ϵ such that the SC system with chain length L and coupling width w of (d_l, d_r, d_g) MN codes over $\mathrm{GEC}(\epsilon)$ converges to zero. In precise, $\epsilon^{\mathrm{BP}}(d_l, d_r, dg, L, w)$ is

$$\sup\{\epsilon \in [0,1] \mid \lim_{t\to\infty} x_i^{(t)} = 0 \text{ for } i = 0,1,\dots,L-1\}.$$

Then, from (II-E) and Theorem 3 and 1 we have the following theorem.

Theorem 3. For any $\mathrm{GEC}(\phi(x;\epsilon))$, the potential threshold ϵ^* of (d_l, d_r, d_g) MN codes is equal to $\epsilon^{\mathrm{SIR}}(\frac{d_r}{d_l})$ for any $d_l > d_r$, $(d_r, d_g) = (2,2)$ and $(3,3)$.

$$
\begin{aligned}
\lim_{w\to\infty}\lim_{L\to\infty} \epsilon^{\mathrm{BP}}(d_l, d_r, dg, L, w) &= \epsilon^{\mathrm{SIR}}(d_r/d_l), \\
\lim_{w\to\infty}\lim_{L\to\infty} R^{\mathrm{MN}}(d_l, d_r, d_g, L, w) &= \frac{d_r}{d_l}.
\end{aligned}
$$

In words, some SC-MN codes universally achieve the SIR limit of any $\mathrm{GEC}(\phi(x;\epsilon))$ in the limit of large L and w.

IV. Conclusion and Future Work

We have shown that some SC-MN codes achieve the SIR limit of any $\mathrm{GEC}(\phi(x;\epsilon))$ via potential function. The future works include an extension erasure multi-acess channels [19] and to more general channels, e.g., PR2 channels with Gaussian noise.

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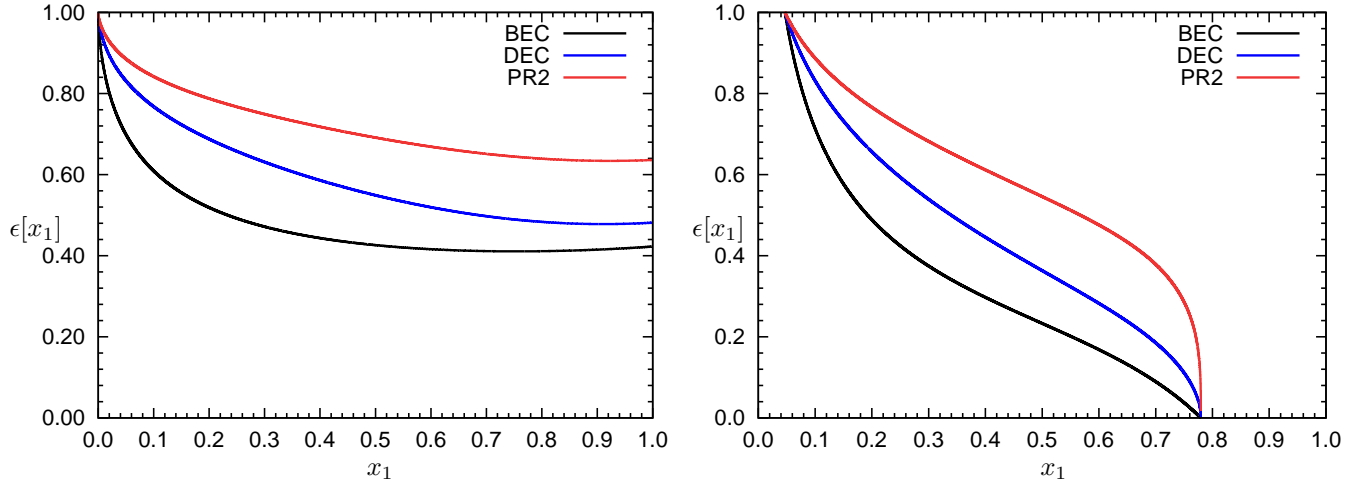## Appendix

In Appendix, we give three GECs as example.

Fig. 2. Plot of $\epsilon[x_1]$ at non-trivial fixed points of $(4, 2, 2)$ MN codes and $(6, 3, 3)$ MN codes for 3 types of generalized erasure channels BEC($\epsilon[x_1]$), DEC($\epsilon[x_1]$), PR2($\epsilon[x_1]$) and PR3($\epsilon[x_1]$).

Example 1 (Binary Erasure Channel). For the binary erasure channel BEC(ϵ) with erasure probability ϵ, $\phi(x; \epsilon), \Phi(x; \epsilon), I(\epsilon)$ and $\epsilon[x_1]$ are respectively given as

$$\phi_{\mathrm{BEC}}(x; \epsilon) = \epsilon,$$
$$\Phi_{\mathrm{BEC}}(x; \epsilon) = \epsilon x,$$
$$I_{\mathrm{BEC}}(\epsilon) = 1 - \epsilon,$$
$$\epsilon[x_1] = \epsilon_{\mathrm{BEC}}[x_1],$$

where $\epsilon_{\mathrm{BEC}}[x_1]$ is the unique solution ϵ_{BEC} of the following equation.

$$x_2[x_1] = \phi_{\mathrm{BEC}}(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g}; \epsilon_{\mathrm{BEC}}[x_1])$$
$$\cdot \{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g - 1}.$$

$\epsilon_{\mathrm{BEC}}[x_1]$ can be written in an explicit form (III-B).

The potential function is given as

$$U_{\mathrm{BEC}}(x_1, x_2[x_1]; \epsilon[x_1]) \tag{17}$$
$$:= 1 - \epsilon_{\mathrm{BEC}}[x_1] \cdot (\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g})$$
$$- \frac{d_r}{d_l}\{1 - (1 - x_1)^{d_r - 1}(1 - x_2[x_1])^{d_g}\}^{d_l}$$
$$- (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g}$$
$$\cdot \left(1 + \frac{d_r x_1}{1 - x_1} + \frac{d_g x_2[x_1]}{1 - x_2[x_1]}\right).$$

Example 2 (Dicode Erasure Channel). For the dicode erasure channel DEC(ϵ) with erasure probability ϵ, the output $\underline{y} \in \{0, 1, -1, ?\}^n$ for given $\underline{x} \in \{0, 1\}^n$ is defined as follows.

$$y_j = \begin{cases} x_j - x_{j-1} & \text{w.p. } 1 - \epsilon, \\ ? & \text{w.p. } \epsilon, \end{cases}$$

where $x_0 := 0$. $\phi(x; \epsilon), \Phi(x; \epsilon), I(\epsilon)$ and $\epsilon[x_1]$ are respectively

given as

$$\phi_{\mathrm{DEC}}(x; \epsilon) = \frac{4\epsilon^2}{(2 - x(1 - \epsilon))^2}$$
$$\Phi_{\mathrm{DEC}}(x; \epsilon) = \frac{4\epsilon^2}{(1 - \epsilon)(2 - (1 - \epsilon)x)},$$
$$I_{\mathrm{DEC}}(\epsilon) = 1 - \frac{2\epsilon^2}{1 + \epsilon},$$
$$\epsilon[x_1] = \epsilon_{\mathrm{DEC}}[x_1],$$

where $\epsilon_{\mathrm{DEC}}[x_1]$ is the unique solution ϵ_{DEC} of the following equation.

$$x_2[x_1] = \phi_{\mathrm{DEC}}(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g}; \epsilon_{\mathrm{DEC}})$$
$$\cdot \{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g - 1}.$$

In the case of DEC(ϵ), $\epsilon_{\mathrm{DEC}}[x_1]$ can be written in an explicit form.

$$\epsilon_{\mathrm{DEC}}[x_1] := \frac{(2 - \psi[x_1])\left(\frac{x_2[x_1]}{\psi[x_1]^{(d_g - 1)/d_g}}\right)^{1/2}}{2 - \psi[x_1]\left(\frac{x_2[x_1]}{\psi[x_1]^{(d_g - 1)/d_g}}\right)^{1/2}}$$

In general, we do not need the explicit form of $\epsilon[x_1]$ for the purpose of this paper. The potential function is given as follows.

$$U_{\mathrm{DEC}}(x_1, x_2[x_1]; \epsilon[x_1])$$
$$:= 1 - \Phi_{\mathrm{DEC}}(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g}; \epsilon_{\mathrm{DEC}}[x_1])$$
$$- \frac{d_r}{d_l}\{1 - (1 - x_1)^{d_r - 1}(1 - x_2[x_1])^{d_g}\}^{d_l}$$
$$- (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g}$$
$$\cdot \left(1 + \frac{d_r x_1}{1 - x_1} + \frac{d_g x_2[x_1]}{1 - x_2[x_1]}\right).$$

Example 3 (Partial Response 2 Channel). For the PR2 channel PR2(ϵ) with erasure probability ϵ, the output $\underline{y} \in$

$\{0, 1, 2, 3, 4, ?\}^n$ for given $\underline{x} \in \{0, 1\}^n$ is defined as follows.

$$y_j = \begin{cases} x_j + 2x_{j-1} + x_{j-2} & \text{w.p. } 1 - \epsilon, \\ ? & \text{w.p. } \epsilon, \end{cases}$$

where $x_0 := 0$. $\phi(x; \epsilon), \Phi(x; \epsilon), I(\epsilon)$ and $\epsilon[x_1]$ are respectively given as

$$\phi_{\text{PR2}}(x; \epsilon) = \frac{4\epsilon^3 \left(4 - 4(1 - \epsilon)x + (1 - \epsilon)x^2\right)}{(4 - 2(1 - \epsilon^2)x - (1 - \epsilon)\epsilon^2 x^2)^2}$$

$$\Phi_{\text{PR2}}(x; \epsilon) = 4\epsilon\left(\frac{1}{2} - \frac{2 - x}{4 - 2(1 - \epsilon^2)x - (1 - \epsilon)\epsilon^2 x^2}\right),$$

$$I_{\text{PR2}}(\epsilon) = 1 - \frac{2\epsilon^3(\epsilon + 1)}{\epsilon^3 + \epsilon_2 + 2},$$

$$\epsilon[x_1] = \epsilon_{\text{PR2}}[x_1],$$

where $\epsilon_{\text{PR2}}[x_1]$ is the unique solution ϵ_{PR2} of the following equation.

$$x_2[x_1] = \phi_{\text{PR2}}(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g}; \epsilon_{\text{PR2}})$$
$$\cdot \{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g - 1}.$$

The potential function is given as follows.

$$U_{\text{PR2}}(x_1, x_2[x_1]; \epsilon[x_1])$$
$$:= 1 - \Phi_{\text{PR2}}(\{1 - (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g - 1}\}^{d_g}; \epsilon_{\text{PR2}}[x_1])$$

$$- \frac{d_r}{d_l}\{1 - (1 - x_1)^{d_r - 1}(1 - x_2[x_1])^{d_g}\}^{d_l}$$
$$- (1 - x_1)^{d_r}(1 - x_2[x_1])^{d_g}$$
$$\cdot \left(1 + \frac{d_r x_1}{1 - x_1} + \frac{d_g x_2[x_1]}{1 - x_2[x_1]}\right).$$